MOND–a pedagogical review[1]

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ABSTRACT

An account is given of the development, and the status, of the modified dynamics (MOND)–a proposed alternative to dark matter, which posits a breakdown of Newtonian dynamics in the limit of small accelerations.

Subject headings: The dark matter problem; galaxy dynamics

1. introduction

The evidence for dark matter is only indirect. What the evidence points to directly is a mass discrepancy in galaxies and other galactic systems: When we count the mass of baryonic matter in such systems–in stars, neutral and high-T gas, etc.–the total sum does not provide enough gravity to explain the observed accelerations in such systems within standard physics. If we adhere to standard dynamics, the need for dark matter is the only solution we can conceive. It is, however, possible that the laws of dynamics, proven in the laboratory and the solar system, cannot be simply applied in the realm of the galaxies. An appropriate modification of the laws of dynamics for parameters that are pertinent to these, might obviate the need for dark matter altogether, if it produces the observed accelerations with only the observed baryonic mass distribution.

But, exactly which system attribute makes the difference? Galactic systems have masses, sizes, and angular momenta that are many orders of magnitude larger than those in the solar system. The large distances involved is a natural culprit. Indeed, there were attempts to modify the distance dependence of gravity: the gravitational force is still taken as proportional to the two masses involved but the decline at large distances is not as strong as in the r^{-2} law. Such a modification cannot, however, explain away dark matter. If the

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modified law is to produce asymptotically flat rotation curves of disc galaxies, as observed, it automatically predicts the wrong form of the mass velocity relation: it gives $M \propto V^2$, instead of $M \propto V^\alpha$, with $\alpha \sim 4$, as required by the observed Tully-Fisher relation (Milgrom 1983). In even more blatant conflict with observations, such modifications predict that the mass discrepancy should increase systematically with system size. In contrast, dwarf spheroidal galaxies, among the smallest in the galactic menagerie, show very large mass discrepancies, much larger then some large galaxies. And, the much larger galaxy clusters show only moderate mass discrepancies. A semi-schematic depiction of the systematics of the mass discrepancy with distance can be seen in Fig 1 in Milgrom (1998), where it is obvious that the observed mass discrepancy does not increase systematically with size.

In the early 1980s I proposed a modified-dynamics based on the acceleration as the relevant system parameter, based on the fact that typical accelerations in galactic systems are many orders of magnitude smaller than those encountered in the solar system. Since then, a handful of us have been working on the development of this scheme, which has involved devising more refined theories, elaborating the observational consequences, and testing them against the data.

2. The modified dynamics

This modified dynamics, MOND, introduces a constant with the dimensions of an acceleration, a_0, and posits that standard Newtonian dynamics is a good approximation only for accelerations that are much larger than a_0. The exact behavior in the opposite limit is described by the specific underlying theory, to be described below. However, the basic point of MOND, from which follow most of the main predictions, can be simply put as follows: a test particle at a distance r from a large mass M is subject to the acceleration a given by

$$a^2/a_0 = MGr^{-2}, \tag{1}$$

when $a \ll a_0$, instead of the standard expression $a = MGr^{-2}$, which holds when $a \gg a_0$. The two expressions may be interpolated to give the heuristic relation

$$\mu(a/a_0)a = MGr^{-2} = a_N, \tag{2}$$

where a_N is the Newtonian expression for the acceleration, and the interpolating function $\mu(x)$ satisfies $\mu(x) \approx 1$ when $x \gg 1$, and $\mu(x) \approx x$ when $x \ll 1$. This expression, while lacking from the formal point of view, is very transparent, and captures the essence of

MOND. I shall describe below more presentable theories based on this basic relation, but these are still phenomenological theories into which the form of $\mu(x)$ has to be put in by hand. It will hopefully follow one day from a more basic underlying theory for MOND, which we still lack. Most of the implications of MOND do not depend strongly on the exact form of μ. Much of the phenomenology pertinent to the mass discrepancy in galactic systems occurs in the deep-MOND regime ($a \ll a_0$), anyway, where we know that $\mu(x) \approx x$.

3. MOND phenomenology

One immediate result of eqs.(1)(2) is that at a large radius around a mass M, the orbital speed on a circular orbit becomes independent of radius. This indeed was a guiding principle in the construction of MOND, which took asymptotic flatness of galaxy rotation curves as an axiom (even though at the time it was not clear how definite, and how universal, this is). Second, this asymptotic rotational speed depends only on the total mass M via $V^4 = MGa_0$. This, according to MOND, is the fact underlying the observed Tully-Fisher-type relations, by which the typical (mean) rotational velocity, V, in a disc galaxy is strongly correlated with the total luminosity of the galaxy, L, in a relation of the form $L \propto V^\alpha$. The power α is around 3-4, and depends on the wavelength band at which L is measured. The close agreement between this TF relation and the prediction of MOND is encouraging; but, to test MOND more precisely on this count, one would have to bridge properly the mass-asymptotic-velocity MOND relation with the commonly presented luminosity-bulk-velocity TF relation. One should use the luminosity in a band where it is a good representative of the stellar mass, take into account not only the stellar mass, as represented by the luminosity, but also the contribution of gas to the mass, and use the asymptotic velocity, as opposed to other measures of the rotational velocity. It has emerged recently (see Verheijen 2001 and reference therein) that if one does all this one indeed obtains a tight and accurate relation of the form predicted by MOND.

But, by far, the most clear-cut test of MOND is provided by disc-galaxy rotation curves, simply because the astronomical observations, and their interpretation, are the most complete and best understood, if still not perfect. What we typically need to know of a galaxy in order to apply this test has been discussed by the various authors who conducted the test; for example, Begeman & al (1991), Sanders & Verheijen (1998), Sanders 1996, de Blok & McGaugh (1998). On the whole, these tests speak cogently for MOND. These test involves fitting the observed rotation curve of a galaxy by that predicted by MOND. Such fits involve one free parameter per galaxy, which is the assumed

conversion factor from luminosity to mass in stars, the so-called mass-to-light ratio. In fact, however, this parameter is not totally free. It is constrained to an extent by what theoretical understanding of galaxy composition tell us. Sanders & Verheijen (1998) who have conducted a MOND rotation-curve analysis of a sample of disc galaxies in the Ursa Major cluster, have compared their deduced MOND best-fit M/L values with theoretical results from stellar-population synthesis. They found a very good agreement. This shows that, to some extent, the MOND rotation curves might be looked at as definite prediction of MOND, which use theoretical M/L values, and not as fits involving one free parameter.

Regarding galactic systems other than galaxies, the comparison of the systematics of the observed mass discrepancy with the expectations from MOND are shown in Figure 2 in Milgrom (1998) based on analyzes referenced there. The agreement is uniform, with one exception: The cores of rich x-ray clusters of galaxies show a considerable mass discrepancy, while, according to MOND there shouldn't be any, because the accelerations there are only of the order of a_0, and not much smaller. (Application of MOND to the clusters at large, say within a few megaparsecs of the center, does predict correctly the mass discrepancy.) The resolution, by MOND, will have to be that these cores harbor large quantities of still undetected baryonic matter, perhaps in the form of dim stars, perhaps as warm gas. The environment, and history, of these cores is so unlike others that this would not be surprising.

In order to appreciate the message that the phenomenological success of MOND carries, we should note the following. According to MOND, the acceleration constant a_0 appears in many independent roles in the phenomenology of the mass discrepancy. For example, in galaxies that have high central accelerations, the mass discrepancy appears only beyond a certain radius; according to MOND , the acceleration at this radius should always be a_0. a_0 also appears as the boundary acceleration between so called high-surface-brightness galaxies (=high acceleration galaxies) which do not show a mass discrepancy near the center, and low-surface-brightness galaxies, where the discrepancy prevails everywhere. a_0 appears in the relation between the asymptotic rotational velocity of a galaxy and its total mass, and in the mass-velocity relation for all sub-a_0 systems, etc., etc..

These roles that a_0 plays are independent in the sense that in the framework of the dark matter paradigm, it is easy to envisage baryon-plus-dark-matter galactic systems that evince any of these appearances of a_0 without showing the others. In other words, in the dark matter paradigm one role of a_0 in the phenomenology does not follow from the others.

This is similar, for example, to the appearances of the Planck constant in different quantum phenomena: in the black-body spectrum, in the photoelectric effect, in the hydrogen spectrum, in superconductivity, etc.. Phenomenologically, these roles of the same constant seem totally unrelated. The only unifying frame is a theory: non-relativistic

quantum mechanics, in this case. MOND is, likewise, a theory that in one fell swoop unifies all the above appearances of a_0 in the phenomenology of galaxy dynamics.

And finally, let me point out a possibly very significant coincidence: The value of the acceleration constant a_0 that fits all the data discussed above is about $10^{-8} cm\ s^{-2}$. This value of a_0 is of the order of some acceleration constants of cosmological significance. It is of the same order as $a_{ex} \equiv cH_0$, where H_0 is the Hubble constant; and, it is also of the order of $a_{cc} \equiv c(\Lambda/3)^{1/2}$, where Λ is the emerging value of the cosmological constant (or "dark energy"). So, for example, a body accelerating at a_0 from rest will approach the speed of light in the life time of the universe.

Because the cosmological state of the universe changes, such a connection, if it is a lasting one, may imply that galaxy evolution does not occur in isolation, affected only by nearby objects, but is, in fact, responding constantly to changes in the state of the universe at large. For example, if the connection of a_0 with the Hubble constant always holds, the changing of the Hubble constant would imply that a_0 must change over cosmic times, and with it the appearance of galactic systems, whose dynamics a_0 controls. If, on the other hand, a_0 is a reflection of a true cosmological constant, then is might be a veritable constant.

4. MOND as an effective theory

But, on the more fundamental side, the above proximity may hint at a deep connection between cosmology and local dynamics in systems that are very small on cosmological scales. Either cosmology somehow enters and affects local laws of physics, such as the law of inertia or the law of gravity, or a common agent affects both cosmology and local physics so as to leave on them the same imprint. This would mean that MOND–and perhaps more cherished notions, such as inertia–is a derived concept, or an effective theory as we would say nowadays. An observed relation between seemingly unrelated constants appearing in a theory (in our case, a_0, the speed of light, and the radius of the horizon) may indicate that it is only an approximation of a theory at a deeper stratum, in which some of the constants do not really have any special role. A parable will help clarify the point: In experiments and observations confined to the vicinity of the earth surface, there appears a constant: the free-fall acceleration, g. If, for some reason, we were restricted to such an ant world (for example because the earth is ever clothed in a thick layer of clouds) unaware of planetary motions, universal gravity, etc., we would have looked on g as a true constant of nature. We would also notice a mysterious relation between this acceleration and two other important

constants: the escape speed c_e (objects thrown with a higher velocity never return) and the radius of the earth $R_\oplus$. This relation: $g = c_e^2/2R_\oplus$, is practically the same as that between a_0, the speed of light, and the Hubble radius, in MOND. But, we do see beyond the earth's surface, and we do know about universal gravity, which tells us that the "constants" g and c_e actually derive from the mass and radius of the earth (hence the relation between the three). They are useful parameters when describing near-earth-surface phenomena, but quite useless in most other circumstances. In a similar vein, a_0 might turn out to be a derived constant, perhaps variable on cosmic time scales, perhaps even of no significance beyond the non-relativistic regime, where MOND has been applied so far. Its connection with the speed of light and the radius of the universe will, hopefully, follow naturally in the underlying theory that still eludes us.

Many instances of such effective theories are known. Even General Relativity is now thought to be an effective, low-energy approximation of a "higher" theory (e.g. a string-inspired theory); an idea that has been anticipated by Sakharov's "induced gravity" idea.

5. Interpretations

Equations(1)(2) have the form of a modification of the law of inertia, but since they are algebraic relations between the MOND and Newtonian accelerations they can simply be inverted to read $a = F/m = a_N f(a_N/a_0)$, which seems to leave the second law intact, while modifying the Newtonian gravitational force ma_N to the MOND value ma. Because gravitation is the sole force that governs galactic dynamics–the only corner where the mass discrepancy has been clearly observed–existing phenomenology does not distinguish well between the interpretations of MOND as modified gravity, and modified inertia. Although there are matter of principle differences between the two interpretations (see below) they pertain to observations that are not yet available. For now we must then investigate both options.

But what exactly is meant by modifying gravity, or modifying inertia? When dealing with pure gravity the distinction is not always clear. For example, the Brans-Dicke theory may be viewed as either. But when other interactions are involved, the distinction is clear. Obviously, modified inertia will enter the dynamics of systems even when gravity is negligible, unlike the case for modified gravity. Formally, the distinction might be made as follows. In a theory governed by an action principle we distinguish three part in the action: The pure gravitational part (for example, the Einstein-Hilbert action in GR), the free action

of the matter degrees of freedom (in GR it also encapsules their interaction with gravity), and the action of interactions between matter degrees of freedom. By ”modifying gravity” I mean modifying the pure-gravity action; by ”modifying inertia” I mean modifying the kinetic (”free”) matter actions.

To understand this definition consider that inertia is what endows the motion of physical objects (particles, fields, large bodies, etc.) with energy and momentum–a currency in the physical world. Motion itself is only of a descriptive value; inertia puts a cost on it. For each kind of object it tells us how much energy and momentum we have to invest, or take away, to change its state of motion by so much. This information is encapsuled in the kinetic action.

For example, take the non-relativistic action for a system of particles interacting through gravity.

$$S = S_\phi + S_k + S_{in} = -(8\pi G)^{-1}\int d^3r\ (\vec\nabla\phi)^2 + \sum_i (1/2)m_i \int dt\ v_i^2 - \int d^3r\ \rho(\vec r)\phi(\vec r), \quad (3)$$

where $\rho(\vec r) = \sum_i m_i \delta(\vec r - \vec r_i)$. (In GR, S_k and S_{in} are lumped together into the particle kinetic action.)

Here, modifying gravity would mean modifying S_ϕ, while modifying inertia would entail changing S_k.

6. MOND as modified gravity

An implementation of MOND as a non-relativistic modified gravity was discussed by Bekenstein & Milgrom (1984), who replaced the standard Poisson action S_ϕ in eq.(3) by an action of the form

$$S_\phi = -(8\pi G)^{-1} a_0^2 \int d^3r\ F[(\vec\nabla\phi)^2/a_0^2]. \quad (4)$$

This gives, upon variation on ϕ, the equation

$$\vec\nabla\cdot[\mu(|\vec\nabla\phi|)\vec\nabla\phi] = 4\pi G\rho(\vec r), \quad (5)$$

where $\mu(x) \equiv dF(y)/dy|_{y=x^2}$. This theory, since it is derived from an action that has all the usual symmetries, satisfies all the standard conservation laws. Its various implications

have been discussed in Bekenstein & Milgrom (1984), Milgrom (1986), Milgrom (1997), and others.

One important point to note is that this theory gives the desired center-of-mass motion of composite systems: Stars, star clusters, etc. moving in a galaxy with a low center-of-mass acceleration are made of constituents whose internal accelerations are much higher than a_0. If we look at individual constituents we see bodies whose total accelerations are high and so whose overall motion is very nearly Newtonian. Yet, their motion should somehow combine to give a MOND motion for the center of mass. This is satisfied in the above theory as shown in Bekenstein & Milgrom (1984). (A similar situation exists in GR: imagine a system made of very tightly bound black holes moving in the weak field of a galaxy, say. While the motions of the individual components is highly relativistic, governed by a non-linear theory, we know that these motions combine to give a simple Newtonian motion for the center of mass.)

This field equation, generically, requires numerical solution, but it is straightforward to solve in cases of high symmetry (spherical, cylindrical, or planar symmetry), where the application of the Gauss law to eq.(5) gives the exact algebraic relation between the MOND ($\vec{g} = -\vec{\nabla}\phi$) and Newtonian ($\vec{g}_N = -\vec{\nabla}\phi_N$) acceleration fields:

$$\mu(g/a_0)\vec{g} = \vec{g}_N, \tag{6}$$

which is identical to the heuristic MOND relation we started with. Note that in general, for configurations of lower symmetry, this algebraic relation does not hold (and, in general, $\vec{g}$ and $\vec{g}_N$ are not even parallel).

It is worth pointing out that in such a modified-gravity theory, the deep-MOND limit corresponds to a theory that is conformally invariant, as discussed in Milgrom (1997). Whether this has some fundamental bearings is not clear, but it does make MOND unique, and enables one to derive useful analytic results, such as an expression for the two-body force, and a virial relation, despite the obstacle of nonlinearity.

There is a large number of physical phenomena that are governed by an equation like eq.(5), each with its own form of the function $\mu(x)$, as detailed in Milgrom (1997), or Milgrom (2001). I would like to concentrate here on one, in particular. It is well known that a stationary, potential flow is described by the Poisson equation: If the velocity field $\vec{u}(\vec{r})$ is derived from a potential, $\vec{u} = \vec{\nabla}\phi$, then the continuity equation, which here determines the flow, reads $\vec{\nabla} \cdot \vec{\nabla}\phi = s(\vec{r})/\varrho_0$, where $s(\vec{r})$ is the source density, and ϱ_0 is the (constant) density of the fluid. When the fluid is compressible, but still irrotational, and barotropic [i.e. has an equation of state of the form $p = p(\rho)$] the stationary flow is described by the

nonlinear Poisson equation. The Euler equation reduces to Bernoulli's law

$$h(\varrho) = -u^2/2 + const., \tag{7}$$

where $dh/d\rho \equiv \rho^{-1}dp/d\rho$. This tell us that ϱ is a function of $u = |\vec{\nabla}\phi|$. Substituting this in the continuity equation gives

$$\vec{\nabla} \cdot [\varrho(|\vec{\nabla}\phi|)\vec{\nabla}\phi] = s(\vec{r}), \tag{8}$$

which has the same form as eq.(5) if we identify ϱ as μ, and the source density s with the normalized gravitational mass density $4\pi G\rho$. Note, however, that from the Bernoulli law, $d\varrho/d|u| = -\varrho|u|/c^2$, where $c^2 = dp/d\varrho$ is the formal squared speed of sound. Thus, in the case of MOND, where we have that μ is an increasing function of its argument, the model fluid has to have a negative compressibility $c^2 < 0$. A cosmological-constant equation of state, $p = -c^2\varrho$, with c the speed of light gives $\varrho(u) = \varrho_0 exp(u^2/2c^2)$, which is not what we need for MOND. The deep-MOND limit, $\mu(u) \approx u/a_0$, corresponds to $p = -(a_0^2/3)\varrho^3$. To get the Newtonian limit at large values of u the equation of state has to become incompressible at some finite density ϱ_0, so that eq.(8) goes to the Poisson equation.

The gravitational force is then the pressure+drage force on sources. For a small (test) static source s, at a position where the fluid speed is $\vec{u}$, the source imparts momentum to the flow at a rate $s\vec{u}$, and so is subject to a force $-s\vec{u}$. The force between sources of the same sign is attractive, as befits gravity.

Note that in such a picture the fluid density itself ϱ does not contribute to the sources of the potential equation, so it does not, itself, gravitate. Also note that, because $\rho = p = 0$ for $\vec{u} = 0$, the fluid behaves as if it has no existence without the sources (masses) that induce velocities in it. Obviously, this picture is anything but directly applicable as an explanation of Newtonian gravity. For example, it is not clear how to obtain the barotropic equation of state that is needed to reproduce MOND. In particular, how does the infinite compressibility appear at a finite critical density, and what is the meaning of this density? Is this due to some phase transition? What happens at densities higher than this critical density? are they accessible at all? Also, there seem to be a drag force on moving sources. In the context of a time-dependent configuration the above equation of state is problematic.

7. MOND as modified inertia

Most people seem to prefer modifying gravity to modifying inertia; perhaps because the latter seems to be less drastic; perhaps because it is a game that has been much played before. I personally feel, without concrete evidence, that there is more potential in modified inertia as the basis for MOND.

Remember first that Newtonian inertia has not been immune to changes. A familiar modification of Newtonian inertia, which is taken to be "nature given", is that brought about by Special Relativity. The single-particle kinetic action in eq.(3) is replaced by $-mc^2 \int \, dt \, [1-(v/c)^2]^{1/2}$, which gives an equation of motion

$$\vec{F} = md(\gamma\vec{v})/dt = m\gamma[\vec{a} + \gamma^2\vec{v}(\vec{v}\cdot\vec{a})/c^2], \tag{9}$$

where γ is the Lorentz factor.

And, physics is replete with instances of modified, acquired, or effective inertia. Electrons and holes in solids can sometimes be described as having a greatly modified mass tensor. Mass renormalization and the Higgs mechanism, modify particle masses and/or endow them with mass: an effective, approximate description that encapsules the effects of interactions of the particles, with vacuum fields in the former instance, and with the Higgs field in the latter. The effects of a fluid on a body embedded in it may sometimes be described as a contribution to to the mass tensor of the body, because its motion induces motion in the fluid which carries energy and momentum. So, modified inertia might also well lie in the basis of MOND.

As a first stage of looking for Mondified inertia it might behoove us to study non relativistic modifications of inertia that incorporate the basic principle of MOND. We seek to modify the particle kinetic action S_k in eq.(3) into an action of the form $S_k[\vec{r}(t), a_0]$, which is a functional of the particle trajectory $\vec{r}(t)$ and depends also on one constant, a_0. It should satisfy the following asymptotic requirements: In the formal limit $a_0 \rightarrow 0$–corresponding to all acceleration measures in the system being much larger than the actual value of a_0 (this is similar to obtaining the classical limit of quantum mechanics by taking the formal limit $\hbar \rightarrow 0$)–it should go into the standard Newtonian action. If we want to retain the MOND phenomenology, according to which in the deep MOND limit G and a_0 appear only through their product Ga_0, then, in the limit $a_0 \rightarrow \infty$, $S_k \propto a_0^{-1}$. This can be seen by rescaling ϕ into ϕ/G in eq.(3) (and deviding the action by G).

The theory should also satisfy the more subtle requirement of the correct center-of-mass motion discuss in the previous section.

General properties of such theories are discussed in detail in Milgrom (1994). Here I summarize, very succinctly, some of the main conclusions.

If the particle free action enjoys the usual symmetries: translational, rotational, and Galilei invariance, than to satisfy the two limits in a_0 it must be non-local. This means that the action cannot be written as $\int L\, dt$, where L is a function of a finite number of derivatives of $\vec{r}(t)$. This might look like a disadvantage, but, in fact, it is a blessing. A local action for MOND would have had to be a higher-derivative theory, and, as such, it would have suffered from the several severe problems that beset such theories. A non-local theory need not suffer from these. Indeed, I have discussed examples that are free of these problems. A non-local action is also a more natural candidate for an effective theory.

While nonlocal theories tend to be rather unwieldy, they do lend themselves to a straightforward treatment of the important issue of rotation curves. This is done via a virial relation that physical, bound trajectories can be shown to satisfy:

$$2S_k[\vec{r}(t), a_0] - a_0\frac{\partial S_k}{\partial a_0} =< \vec{r}\cdot\vec{\nabla}\phi >, \tag{10}$$

where ϕ is the (unmodified) potential in which the particle is moving, $<>$ marks the time average over the trajectory, and S_k is the value of the action calculated for the particular trajectory (S_k is normalized to have dimensions of velocity square). In the Newtonian case this reduces to the usual virial relation. Applying this relation to circular orbits in an axi-symmetric potential, and noting that, on dimensional grounds, on such orbits with radius r and velocity v we must have $S_k(r, v, a_0) = v^2\mu(v^2/ra_0)$, we end up with the expression for the velocity curve

$$(v^2/r)\mu(v^2/ra_0) = d\phi/dr. \tag{11}$$

Thus the algebraic relation that was first used in MOND as a naive application of eq.(2), and which all existing rotation-curve analyzes use, is exact in modified-inertia MOND. In modified gravity this expression is a only good approximation.

Another important difference between the two interpretations is worth noting. Unlike (non-relativistic) modified gravity, where the gravitational field is modified, but in it all bodies at the same position undergo the same acceleration, in modified inertia the acceleration depends not only on position, but also on the trajectory. In the case of SR the acceleration depends on the velocity as well, but in more general theories it might depend on other properties of the orbit. There is still a generalized momentum whose rate of change *is* a function of position only ($m\gamma\vec{v}$ in SR) but this rate is not the acceleration. This larger freedom in modified inertia comes about because we implement the modification via a modification of the action as a functional of the trajectory; namely, a function of an infinite number of variables; so, different trajectories might suffer different modifications.

In modifying gravity we modify one function of the three coordinates (the gravitational potential). This is an obvious point, but is worth making because in interpreting data we equate observed accelerations with the gravitational field. While this is still true in modified gravity it is not so in modified inertia.

We can exemplify this point by considering the claimed anomaly in the motions of the Pioneer 10 and 11 spacecraft. Analysis of their motion have shown an unexplained effect (see Anderson & al 2001) that can be interpreted as being due to an unexplained constant acceleration towards the sun of about $7 \cdot 10^{-8} cm\ s^{-2}$, of the order of a_0. This might well be due to some systematic error, and not to new physics. This suspicion is strengthened by the fact that an addition of a constant acceleration of the above magnitude to the solar gravitational field is inconsistent with the observed planetary motions (e.g. it gives a much too large rate of planetary perihelion precession).

MOND could naturally explain such an anomalous acceleration: We are dealing here with the strongly Newtonian limit of MOND, for which we would have to know the behavior of the extrapolating function $\mu(x)$ at $x >>> 1$, where $\mu \approx 1$. We cannot learn about this from galaxy dynamics, so we just parameterize μ in this region: $\mu \approx 1 - \xi x^{-n}$. (This is not the most general form; e.g. μ may approach 1 non analytically in x^{-1}, for example as $1 - exp(-\xi x)$.) Be that as it may, if $n = 1$ we get just the desired effect in MOND: the acceleration in the field of the sun becomes $M_\odot G r^{-2} + \xi a_0$ in the sun's direction. As I said above, in a modified gravity interpretation this would conflict with the observed planetary motions; but, in the modified-inertia approach it is not necessarily so. It may well be that the modification enters the Pioneers motion, which corresponds to unbound, hyperbolic motions, and the motion of bound, and quasi-circular trajectories in a different way. For example, the effective μ functions that correspond to these two motions might have different asymptotic powers n.

8. vacuum effects and MOND inertia

Because MOND revolves around acceleration, which is so much in the heart of inertia, one is directed, with the above imagery in mind, to consider that inertia itself, not just MOND, is a derived concept reflecting the interactions of bodies with some agent in the background. The idea, which is as old as Newton's second law, is the basic premise of the Mach's principle. The great sense that this idea makes has lead many to attempt its implementation. The agent responsible for inertia had been taken to be the totality of matter in the Universe.

Arguably, an even better candidate for the inertia-producing agent, which I have been considering since the early 1990s, in the hope of understanding MOND's origin, is the vacuum. The vacuum is known to be implicated in producing or modifying inertia; for example, through mass renormalization effects, and through its contribution to the free Maxwell action in the form of the Euler-Heisenberg action Itzykson & Zuber 1980. Another type of vacuum contributions to inertia have been discussed by Jaekel & Reynaud 1993. But, it remains moot whether the vacuum can be fully responsible for inertia.

The vacuum is thought to be Lorentz invariant, and so indifferent to motion with constant speed. But acceleration is another matter. As shown by Unruh in the 1970s, an accelerated body is alive to its acceleration with respect to the vacuum, since it finds itself immersed in a telltale radiation, a transmogrification of the vacuum that reflects his accelerated motion. For an observer on a constant-acceleration (a) trajectory this radiation is thermal, with $T = \alpha a$, where $\alpha \equiv \hbar/2\pi kc$. The effect has been also calculated approximately for highly relativistic circular motions; the spectrum is then not exactly thermal. In general, it is expected that the effect is non-local; i.e., depends on the full trajectory.

Unruh's result shows that the vacuum can serve as an inertial frame. But this is only the first step. The remaining big question is how exactly the vacuum might endow bodies with inertia. At any rate, what we want is the full MOND law of inertia, with the transition occurring at accelerations of order a_0 that is related to cosmology. We then have to examine the vacuum in the context of cosmology. How it affects, and is being affected by, cosmology. One possible way in which cosmology might enter is through the Gibbons-Hawking effect, whereby even inertial observers in an expanding universe find themselves embedded in a palpable radiation field that is an incarnation of the vacuum. The problem has been solved for de Sitter Universe, which is characterized by a single constant: the cosmological constant, Λ, which is also the square of the (time independent) Hubble constant. In this case the spectrum is also thermal with a temperature $T = \alpha c(\Lambda/3)^{1/2}$.

In the context of MOND it is interesting to know what sort of radiation an observer sees, who is accelerated in a non-trivial universe: If the Unruh temperature is related to inertia, then it might be revealing to learn how this temperature is affected by cosmology. This can be gotten for the case of a constant-acceleration observer in a de Sitter Universe. For this case the radiation is thermal with a temperature $T = \alpha(a^2 + c^2\Lambda/3)^{1/2}$ Deser & Levin 1997. Inertia, which is related to the departure of the trajectory from that of an inertial observer, who in de Sitter space sees a temperature $\alpha c(\Lambda/3)^{1/2}$, might be proportional to the temperature difference

$$\Delta T = \alpha[(a^2 + c^2\Lambda/3)^{1/2} - c(\Lambda/3)^{1/2}], \tag{12}$$

and this behaves exactly as MOND inertia should: it is proportional to a for $a >> a_0 \equiv 2c(\Lambda/3)^{1/2}$, and to a^2/a_0 for $a << a_0$; and, we reproduce the connection of a_0 with cosmology. Of course, in the modified-inertia paradigm this would reflect on a "linear", constant-acceleration motion, while circular trajectories will probably behave differently. But the emergence of an expression *a-la* MOND in this connection with the vacuum is very interesting.

9. Relativistic theories

We still want a relativistic extension of MOND. Such a theory is needed for conceptual completion of the MOND idea. But, it is doubly needed because we already have observed relativistic phenomena that show mass discrepancies, and we must ascertain that there too the culprit is not dark matter but modified dynamics.

Because of the near values of a_0 and the Hubble acceleration, there are no local black holes that are in the MOND regime. The only system that is strongly general relativistic and in the MOND regime is the Universe at large. This, however means that we would need a relativistic extension of MOND to describe cosmology. In fact, as I have indicated, MOND itself may derive from cosmology, so it is possible that the two problems will have to be tackled together as parts and parcels of a unified concept. And, because the cosmological expansion is strongly coupled with the process of structure formation this too will have to await a modified relativistic dynamics for its treatment.

Several relativistic theories incorporating the MOND principle have been discussed in the literature, but none is wholly satisfactory (see, e.g. Bekenstein & Milgrom (1984), Bekenstein 1988, Sanders 1997, and references therein).

There have also been attempts to supplement MOND with extra assumptions that will enable the study of structure formation, so as to get some glimpse of structure formation in MOND. For these see Milgrom (1989), Sanders (2001), and Nusser (2001).

Gravitational light deflection, and lensing, is another phenomenon that requires modified relativistic dynamics. It is tempting to take as a first approximation the deflection law of post-Newtonian General Relativity with a potential that is the non-relativistic MOND potential (see e.g. analyzes by Qin & al 1995, and Mortlock & Turner 2001 based on this assumption). This, however, is in no way guaranteed. In GR this is only a post-Newtonian approximation, and perhaps it would turn out to be a post-Newtonian approximation of MOND (i.e. an approximation of MOND in the almost Newtonian,

$a >> a_0$ regime). But, there is no reason to assume that it is correct in the deep-MOND regime. Even in the framework of this assumption one needs to exercise care. For example, the thin-lens hypothesis, by which it is a good approximation to assume that all deflecting masses are projected on the same plane perpendicular to the line of sight, breaks down in MOND. For example, n masses, M, arranged along the line of sight (at inter-mass distances larger that the impact parameter) bend light by a factor $n^{1/2}$ more than a single mass nM.

Also note that we may expect surprises in mondified inertia where we cannot even speak of the modified, MOND potential, as alluded to above.

REFERENCES

Anderson, J.D. & al 2001, preprint, gr-qc/ 0104064

Begeman, K.G., Broeils, A.H., & Sanders, R.H. 1991, MNRAS 249, 523

Bekenstein, J. 1988, Phys. Lett. B202, 497

Bekenstein, J. & Milgrom, M. 1984, ApJ 286,7

de Blok, W.J.G. & McGaugh, S.S. 1998, ApJ 508, 132

Itzykson, C & Zuber, J.B. 1980, Quantum Fielf Theory, McGraw-Hill

Deser, S. & Levin, O. 1997 Clas. Quant. Grav. 14, L163

Milgrom, M. 1983, ApJ 270, 371

Milgrom, M. 1986, ApJ 302, 617

Milgrom, M. 1989, Comm. Astrophys. 13:4, 215

Milgrom, M. 1994, Ann. Phys. 1994, 229, 384

Milgrom, M. 1997, Phys. Rev. E 56, 1148

Milgrom, M. 1998, Proceedings of the DARK98 meeting in Heidelberg July 1998, astro-ph/9810302

Milgrom, M. 2001, Forces in nonlinear theories, preprint

Mortlock, D.J., & Turner, E.L. 2001, MNRAS 327, 557

Nusser, A. 2001, astro-ph/0109016, MNRAS in press

Qin, B., Wu, X.P., & Zou, Z.L. 1995, A&A, 296, 264

Jaekel, M.T. & Reynaud, S. 1993, J. de Physique 3, 1093

Sanders, R.H. 1996, ApJ 473, 117

Sanders, R.H. 1997, ApJ 480, 492

Sanders, R.H. 2001, ApJ, in press

Sanders, R.H. 1999, ApJ 512, L23

Sanders, R.H. & Verheijen, M.A.W 1998, ApJ 503, 97

Verheijen, M.A.W 2001, astro-ph/0108225